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Engineering Notes

New Lambert Algorithm Using the Hamilton–Jacobi–Bellman Equation

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I. Introduction

THE two-point boundary-value problem (TPBVP) for Keplerian motion, which is known as Lambert's problem, is a fundamental problem in space trajectory design. Lambert's problem may be stated as follows: given initial and final position vectors, determine the initial velocity which will allow a transfer in a specified flight time. The classical approach to Lambert's problem is based on geometry of the radius vectors, which directly stems from Lambert's theorem [1]. After the early works of Gauss [2], a number of numerical algorithms were proposed by Lancaster et al. [3], Battin [4], and other researchers [5–7]. However, these solution procedures cannot be applied to the perturbed problem, because the geometry of the dynamics changes. On the other hand, general TPBVPs are solved by means of numerical methods such as the shooting method and the finite difference method. Lizia et al. [8] and Armellin and Topputo [9] developed an integration scheme for the solution of the TPBVP and astrodynamics applications concerning the computation of trajectories in the vicinity of the libration points in the restricted three- and four-body problem. Guibout and Scheeres [10] solved the TPBVP using the theory of canonical transformation. These approaches [8–10] rely on a power series expansion along a nominal trajectory.

Hamilton's principle [11] is an alternative formulation of the differential equations of motion of spacecraft, which states that the trajectory between two specified states at two specified times is an extremum of the action integral. Motivated by this observation, this paper attempts to show that a solution to Lambert's problem is directly obtained by minimizing the action integral. This problem can be viewed as an optimal control problem by replacing kinetic energy with a quadratic performance index of the control input, so that the initial velocity is found as the optimal control. Then, the solution is given by the Hamilton–Jacobi–Bellman (HJB) equation. The approximation methods to the solutions to the HJB equation were studied by many authors based on series expansion techniques [12–14]. In [14], a closed-form solution of the HJB equation is obtained by expanding the value function as a power series in terms of the state and the constant Lagrange multipliers. Although higher-order approximations are possible to obtain by series expansion solutions, their computations are time-consuming and there is no guarantee that the resulting solution improve the performance. Another approach is through successive approximation, where the HJB equation is reduced to a sequence of the first-order linear partial differential equations [15–17]. Mizuno and Fujimoto [18] showed that the HJB equation is effectively solved by the Galerkin spectral method with Chebyshev polynomials based on successive approximation. In this paper, the TPBVP of the Hamiltonian system is treated as an optimal control problem where the Lagrangian function plays a role as a performance index. Similar to Mizuno and Fujimoto, our approach is based on the expansion of the value function in the Chebyshev series with unknown coefficients, considering the computational advantages of the use of Chebyshev polynomials. The differential expressions that arise in the HJB equation are also expanded in Chebyshev series with the unknown coefficients. As a consequence, the algorithm is much simpler than the procedure based on series expansion, and higher-order approximations are possible to obtain for more complicated nonlinear dynamics. Our algorithm can provide a solution to the TPVBP using the spectral information about the gravitation potential function and is also applicable to the TPVBP under a higher-order perturbed potential function without any modification.

The paper is organized as follows. Section II is the problem statement. Section III.A reviews the Hamilton principle and the motivation of our method, Sec. III.B introduces the HJB equation in optimal control theory, and Sec. III.C formulates the solution procedure. Section IV presents simulation results for the two-body problem and the circular restricted three-body problem.

II. Problem Statement

Consider a spacecraft subject to the central gravity field. The normalized equation of motion is described as

$$\ddot{\boldsymbol{r}} = -\frac{\boldsymbol{r}}{r^3} \tag{1}$$

where $\boldsymbol{r}$ is the position vector of the satellite from the center of the gravity and $r = |\boldsymbol{r}|$. Now we consider the following problems.

Problem 2: Impulsive Thrust Orbital Transfer (Lambert's Problem): Consider a spacecraft located at $\boldsymbol{r}_0$ at $t = t_0$. Find the initial velocity that is required to transfer to $\boldsymbol{r}_f$ in time t_f.

III. Solving Nonlinear Dynamics Using the Hamilton–Jacobi–Bellman Equation

A. Motivation

In this section we show that Lambert's problem can be viewed as an optimal control problem by replacing kinetic energy with a quadratic performance index of the control input. First, we review Hamilton's principle for conservative systems [11].

Hamilton's Principle:

Let T, V be the kinetic and potential energy of the system, respectively. Then the Lagrangian function is given by $L(\boldsymbol{q}, \dot{\boldsymbol{q}}, t) = T - V$, where $\boldsymbol{q} = [q_1, \cdots, q_n]^T$ is a set of generalized coordinates. Recall that Hamilton's Principle for conservative systems states that the motion is such that the action integral

$$I = \int_{t_1}^{t_2} L \, \mathrm{d}t \tag{2}$$

is an extremum among all possible paths that travel from its position at $t = t_1$ to its position at $t = t_2$; that is, the variation of the line integral I for fixed t_1 and t_2 is zero

$$\delta I = \delta \int_{t_1}^{t_2} L \, \mathrm{d}t = 0 \tag{3}$$

If L is expressed in terms of the Hamiltonian by $H(\boldsymbol{q}, \boldsymbol{p}, t) = \boldsymbol{p}^T \dot{\boldsymbol{q}} - L(\boldsymbol{q}, \dot{\boldsymbol{q}}, t)$, Eq. (2) is then

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$$I[\boldsymbol{q},\boldsymbol{p}] = \int_{t_0}^{t_f} \boldsymbol{p}^T\dot{\boldsymbol{q}} - H(\boldsymbol{q},\boldsymbol{p},t)\,\mathrm{d}t \tag{4}$$

Hamilton's principle directly leads to Hamilton's canonical equations of motion

$$\dot{\boldsymbol{q}} = \frac{\partial H(\boldsymbol{q},\boldsymbol{p},t)}{\partial \boldsymbol{p}} \tag{5}$$

$$\dot{\boldsymbol{p}} = -\frac{\partial H(\boldsymbol{q},\boldsymbol{p},t)}{\partial \boldsymbol{q}} \tag{6}$$

where the generalized coordinates $\boldsymbol{q} = [q_i,\cdots,q_n]^T$ and generalized momenta $\boldsymbol{p} = [p_1,\cdots,p_n]^T$ are considered as the independent variables. We can obtain first-order equations of motion from Hamilton's principle, while Lagrange's formulation leads to second-order equations.

The key idea is the interpretation of the Hamilton's principle as an optimal control problem with nonlinear cost function. Consider the two-body central force problem. From Hamilton's principle, we wish to find two values $\boldsymbol{p}(t_0)$ and $\boldsymbol{p}(t_f)$ such that the line integral of $L = \boldsymbol{q}^T\boldsymbol{p} - H(\boldsymbol{q},\boldsymbol{p},t)$ is an extremum. The Hamiltonian is described by

$$H(\boldsymbol{q},\boldsymbol{p}) = \frac{1}{2}\boldsymbol{p}^T\boldsymbol{p} + U(\boldsymbol{q}) \tag{7}$$

where $U(\boldsymbol{q}) \leq 0$ is a potential function, e.g., $U(\boldsymbol{q}) = -\frac{1}{r}$. Note that

$$L(\boldsymbol{q},\boldsymbol{p},t) \geq 0 \tag{8}$$

for all $\boldsymbol{p}$. Equation (8) implies that if a correct path $\boldsymbol{q}(t), \in [t_0,t_f]$ is given, we wish to find $\boldsymbol{p}(t), \in [t_0,t_f]$ to minimize the action integral Eq. (4). From this observation, Eq. (5) can be viewed as an affine nonlinear system with input $\boldsymbol{u}$, which is described as

$$\dot{\boldsymbol{x}} = \boldsymbol{u} \tag{9}$$

with the replacement

$$\boldsymbol{q} \to \boldsymbol{x}, \qquad \boldsymbol{p} \to \boldsymbol{u}$$
$$\int_{t_0}^{t_f} \frac{1}{2}\boldsymbol{p}^T\boldsymbol{p} - U(\boldsymbol{q})\,\mathrm{d}t \to \int_{t_0}^{t_f} \frac{1}{2}\|\boldsymbol{u}\|^2 + R(\boldsymbol{x})\,\mathrm{d}t$$

where $\boldsymbol{x} \in \mathbb{R}^n$ is a state, $\boldsymbol{u} \in \mathbb{R}^n$ is a control, and $R(\boldsymbol{x}) = -U(\boldsymbol{x}) \geq 0$. Now Lambert's problem is transformed into the nonlinear optimal control problem (problem 1) subject to fixed-state two-point boundary conditions and cost function Eq. (11).

Problem 1: Consider the system

$$\dot{\boldsymbol{x}} = \boldsymbol{u} \tag{10}$$

Find the control sequence such that the nonlinear performance index

$$J(u;\boldsymbol{x}_0) = \int_{t_0}^{t_f} \frac{1}{2}\|\boldsymbol{u}\|^2 + R(\boldsymbol{x})\,\mathrm{d}t \tag{11}$$

is minimized.

Note that the additional condition $R(\boldsymbol{x}) \geq 0$ is imposed. However, $R(\boldsymbol{x}) = -U(\boldsymbol{x}) \geq 0$ can be satisfied for our problem even if the perturbed potential function is assumed.

B. Optimal Control Theory

In this section, the formulation of the optimal control problem which arises in the previous section is given. It is well known that Pontryagin's Maximum Principle provides the necessary conditions for the optimality [19]. Consider the affine nonlinear system

$$\dot{\boldsymbol{x}} = f(\boldsymbol{x}) + g(\boldsymbol{x})\boldsymbol{u} \tag{12}$$

and initial condition

$$\boldsymbol{x}(t_0) = \boldsymbol{x}_0 \tag{13}$$

where $\boldsymbol{x} \in \mathbb{R}^n$ is a state, $\boldsymbol{u} \in \mathbb{R}^m$ is a control. The performance of the system is measured by the performance index

$$J(\boldsymbol{u};\boldsymbol{x}_0) = \int_{t_0}^{t_f} \frac{1}{2}\|\boldsymbol{u}(t)\|^2 + R(\boldsymbol{x})\,\mathrm{d}t + \phi(\boldsymbol{x}(t_f)) \tag{14}$$

where $R(\boldsymbol{x}) \geq 0, \phi(\boldsymbol{x}(t_f)) \geq 0$ are given scalar nonlinear functions of their arguments. The final time t_f is assumed to be given explicitly. All the functions f, U, and g are assumed to be continuously differentiable in each argument. The objective is to seek $\boldsymbol{u}(t)$; $t \in [t_0,t_f]$ such that J given by Eq. (14) is minimized. Define the pre-Hamiltonian

$$\bar{H}(\boldsymbol{x},\boldsymbol{p},\boldsymbol{u},t) \triangleq \frac{1}{2}\|\boldsymbol{u}(t)\|^2 + R(\boldsymbol{x}) + \boldsymbol{p}^T f(\boldsymbol{x},\boldsymbol{u},t) \tag{15}$$

and set $H(\boldsymbol{x},\boldsymbol{p},t) \triangleq \bar{H}(\boldsymbol{x},\boldsymbol{p},\boldsymbol{u}^*,t)$. Pontryagin's minimum principle states that the optimal control $\boldsymbol{u}^*$ minimize the pre-Hamiltonian

$$\boldsymbol{u}^* = \arg\min_{\boldsymbol{u}} \bar{H}(\boldsymbol{x},\boldsymbol{p},\boldsymbol{u},t) \tag{16}$$

and the optimal state trajectory $\boldsymbol{x}$ and corresponding adjoint vector $\boldsymbol{p}$ must satisfy

$$\dot{\boldsymbol{x}} = \frac{\partial H(\boldsymbol{x},\boldsymbol{p},t)}{\partial \boldsymbol{p}} \tag{17}$$

$$\dot{\boldsymbol{p}} = -\frac{\partial H(\boldsymbol{x},\boldsymbol{p},t)}{\partial \boldsymbol{x}} \tag{18}$$

$$\boldsymbol{p}(t_f) = \frac{\partial \phi(\boldsymbol{x}(t_f),t_f)}{\partial \boldsymbol{x}} \tag{19}$$

In dynamic programming, when an optimal control $\boldsymbol{u}^*$ exists it is given by

$$\boldsymbol{u}^* = -g(\boldsymbol{x})^T\frac{\partial V^*}{\partial \boldsymbol{x}} \tag{20}$$

where V^* is the solution to the HJB equation

$$\frac{\partial V^*}{\partial t} + \min_{\boldsymbol{u}}\left[\frac{\partial V^{*T}}{\partial \boldsymbol{x}}(f(\boldsymbol{x}) + g(\boldsymbol{x})\boldsymbol{u}) + R(\boldsymbol{x}) + \frac{1}{2}\|\boldsymbol{u}(t)\|^2\right] = 0 \tag{21}$$

$$V^*(\boldsymbol{x}(t_f),t_f) = \phi(\boldsymbol{x}(t_f)) \tag{22}$$

Substituting Eq. (20) into Eq. (21), we have

$$\frac{\partial V^*}{\partial t} + \frac{\partial V^{*T}}{\partial x}f + R(\boldsymbol{x}) - \frac{1}{2}\frac{\partial V^{*T}}{\partial \boldsymbol{x}}g(\boldsymbol{x})g(\boldsymbol{x})^T\frac{\partial V^*}{\partial \boldsymbol{x}} = 0 \tag{23}$$

C. Semi-Analytical Approach to the Solution of the Hamiltonian Dynamics

1. *Successive Approximation of the Hamilton–Jacobi–Bellman Equation*

In [16], a method to compute the solution of the HJB equation by computing a sequence of the linear partial differential equations called the generalized HJB (GHJB) equation is introduced for infinite-time horizon problems. In this section, the approximation method of the solution to the nonlinear optimal control problems for the finite time fixed terminal-state problems is derived. That is, the new optimality condition of the fixed teminal-state problem is expressed by the GHJB equation and the condition of the induced Lagrange multiplier. In the following we extend the algorithm [16] to the finite time problem with the terminal constraint

$$\boldsymbol{\psi}(\boldsymbol{x}(t_f)) = \boldsymbol{x}(t_f) - \boldsymbol{x}_f = 0 \tag{24}$$

To obtain the approximate solution of Eq. (23), given an initial control $\boldsymbol{u}^{(0)}$, consider the performance index

$$J^{(k)}(\boldsymbol{u}^{(k)};\boldsymbol{x}_0) = \int_{t_0}^{t_f} \frac{1}{2}\|\boldsymbol{u}^{(k)}(t)\|^2 + R(\boldsymbol{x})\,\mathrm{d}t + \phi(\boldsymbol{x}(t_f)) \tag{25}$$

and the linear partial differential equation

$$\frac{\partial V^{(k)}}{\partial t} + \frac{\partial V^{(k)T}}{\partial \boldsymbol{x}}(f(\boldsymbol{x}) + g(\boldsymbol{x})\boldsymbol{u}^{(k)}) + R(\boldsymbol{x}) + \|\boldsymbol{u}^{(k)}\|^2 = 0 \tag{26}$$

$$V^{(k)}(\boldsymbol{x}(t_f), t_f) = \phi(\boldsymbol{x}(t_f)) \tag{27}$$

which is called the generalized HJB equation (GHJB) in [16]. From Eqs. (26) and (27), we have $V^{(k)}(\boldsymbol{x}_0, t_0) = J^{(k)}(\boldsymbol{u}^{(k)};\boldsymbol{x}_0)$.

For the terminal constraints problem, using the augmented cost function

$$J(\boldsymbol{u};\boldsymbol{x}_0) = \int_{t_0}^{t_f} \frac{1}{2}\|\boldsymbol{u}^{(k)}(t)\|^2 + R(\boldsymbol{x})\,\mathrm{d}t + \boldsymbol{b}^T\boldsymbol{\psi}(\boldsymbol{x}(t_f)) \tag{28}$$

the constrained optimization problems can be translated into unconstrained optimization problems where $\boldsymbol{b}$ is known as the Lagrange multiplier. The problem is now to choose $\boldsymbol{u}(t)$, $t \in [t_0, t_f]$ such that J is minimized and to choose $\boldsymbol{b}$ such that Eq. (28) is satisfied. To obtain a recursive formulation for $\boldsymbol{u}(t)$ and $\boldsymbol{b}$, define

$$J^{(k)}(\boldsymbol{u}^{(k)};\boldsymbol{x}_0) = \int_{t_0}^{t_f} \frac{1}{2}\|\boldsymbol{u}^{(k)}(t)\|^2 + R(\boldsymbol{x})\,\mathrm{d}t + \boldsymbol{b}^{(k)T}\boldsymbol{\psi}(\boldsymbol{x}(t_f)) + \frac{c}{2}\|\boldsymbol{\psi}(\boldsymbol{x}(t_f))\|^2 \tag{29}$$

As with the free terminal-state problem [16], set

$$\phi^{(k)}(\boldsymbol{x}(t_f)) \triangleq \boldsymbol{b}^{(k)T}\boldsymbol{\psi}(\boldsymbol{x}(t_f)) + \frac{c}{2}\|\boldsymbol{\psi}(\boldsymbol{x}(t_f))\|^2$$

Then set the boundary conditions for the GHJB (26)

$$V^{(k)}(t_f) = \phi^{(k)}(\boldsymbol{x}(t_f)) \tag{30}$$

Then select $\boldsymbol{u}^{(k)}$, $\boldsymbol{b}^{(k)}$ as

$$\boldsymbol{u}^{(k)} = -g(\boldsymbol{x})^T \frac{\partial V^{(k-1)}}{\partial \boldsymbol{x}} \tag{31}$$

$$\boldsymbol{b}^{(k)} = \boldsymbol{b}^{(k-1)} + c\boldsymbol{\psi}(\boldsymbol{x}(t_f))^T \tag{32}$$

where $V^{(k-1)}$ is the solution of Eq. (26) for $(k-1)$ step and c is a positive constant that is chosen to satisfy

$$\phi^{(k+1)}(\boldsymbol{x}(t_f)) \le \phi^{(k)}(\boldsymbol{x}(t_f)) \tag{33}$$

It has been shown in [15,16] that the algorithm converges to the optimal control $\boldsymbol{u}^*$ and optimal cost V^*. An extended version of these results appears in Appendix A. The recursive algorithm is summarized below.

[successive approximation]

1) For $k = 0$, define $\boldsymbol{u}^{(0)} = 0$ and $\boldsymbol{b}^{(0)} = 0$.
2) Solve GHJB Eq. (26) to compute $V^{(k)}$.
3) Update $\boldsymbol{u}^{(k)}$ and $\boldsymbol{b}^{(k)}$ by Eqs. (31) and (32), respectively.
4) Set $k = k + 1$ and repeat steps 2 and 3.

By iterating the process, the solution to the GHJB equation uniformly converges to the solution of the HJB equation. However, the GHJB equation has no general closed-form solution. In the next section, we use the Galerkin spectral method to approximate the GHJB equation.

2. *Spectral Method Using Chebyshev Polynomials*

The spectral method is based on the assumption that the solution can be approximated by a sum of $N + 1$ basis functions [20]. In order that the difference between the exact solution and the approximated solution might be identically equal to zero, the problem is to choose the series coefficients so that the difference is minimized. Among many types of basis functions, a natural basis for the approximation of functions on a finite interval employs the Chebyshev polynomials $T_n(x)$ and an approximating expansion of the form

$$V_N(\boldsymbol{x}, t) \triangleq \sum_{i_1=0}^{N}\cdots\sum_{i_n=0}^{N} c^{(k)}_{i_1,\cdots,i_n}(t) T_{i_1}(x_1)\cdots T_{i_n}(x_n) \tag{34}$$

where $c^{(k)}_{i_1,\cdots,i_n}(t)$ represents the time-varying coefficient of Chebyshev polynomial $T_{n_1}(x_1)\cdots T_{i_n}(x_n)$, and N denotes the truncation order of the Chebyshev series. The nth order Chebyshev polynomial is given by

$$\cos n\theta \triangleq T_n(\cos\theta) = T_n(x), \qquad x = \cos\theta, \qquad x \in [-1, 1] \tag{35}$$

Although $T_n(x)$ are polynomials in x, a Chebyshev series is a Fourier cosine expansion with change of variables. Because Chebyshev polynomials are defined on the interval $[-1, 1]$, we must change variables to satisfy the condition. The Chebyshev polynomials form a complete orthogonal system on the interval $[-1, 1]$ with respect to the weighting function $\frac{1}{\sqrt{1-x^2}}$, i.e.,

$$\int_{-1}^{1} \frac{1}{\sqrt{1-x^2}} T_k(x)T_l(x)\,\mathrm{d}x = \begin{cases} \pi & (k = l = 0) \\ \frac{\pi}{2} & (k = l \ne 0) \\ 0 & (k \ne l) \end{cases} \tag{36}$$

Moreover, the Chebyshev series of smooth functions decrease exponentially fast with N; therefore, the approximation converges exponentially fast to the exact solution. The continuous expansion coefficients are found by exploiting the orthogonality Eq. (36)

$$c_n = \frac{2}{\pi}\int_{-1}^{1} f(x)T_n(x)\frac{\mathrm{d}x}{\sqrt{1-x^2}}, \qquad n = 0, 1, \cdots N \tag{37}$$

The important relations of Chebyshev polynomials are summarized in Appendix B. For the one dimensional case ($n = 1$), each term of Eq. (26) can be expanded to

$$\frac{\partial V^{(k)}}{\partial x} \triangleq \sum_{i_1=0}^{N} c'^{(k)}_{i_1}(t)T_{i_1} \tag{38}$$

$$u^{(k)} = \frac{\partial V^{(k-1)}}{\partial x} = \sum_{i_1=0}^{N} c'^{(k-1)}_{i_1}(t)T_{i_1} \tag{39}$$

$$\frac{\partial V^{(k)T}}{\partial x}u^{(k)} \triangleq \sum_{i_1=0}^{N}\left(\sum_{m=0}^{N} P_{i_1,m}c^{(k)}_m(t)\right)T_{i_1} \tag{40}$$

$$U(x) \triangleq \sum_{i_1=0}^{N} l_{i_1}T_{i_1} \tag{41}$$

$$\|u^{(k)}\|^2 = \left(\sum_{i_1=0}^{N} c'^{(k-1)}_{i_1}(t)T_{i_1}\right)^2 \triangleq \sum_{i_1=0}^{N} a_{i_1}(t)T_{i_1} \tag{42}$$

where c'_{i_1} denotes the coefficients of the Chebyshev expansion

$$\sum_{i_1=0}^{N}\frac{\partial}{\partial x_1}(c_{i_1}T_{i_1}) \triangleq \sum_{i_1}^{N} c'_{i_1}T_{i_1} \tag{43}$$

and $P = (P_{i_1,m})$ is a constant matrix that is obtained by matching terms in Eq. (40). Because

$$\int_{-1}^{1}\frac{1}{\sqrt{1-x^2}}\alpha_{n_i,n_i}T_k(x)T_l(x)\,\mathrm{d}x = 0 \Rightarrow \alpha_{n_i,n_i} = 0, \quad \forall\ i$$
$$\int_{-1}^{1}\frac{1}{\sqrt{1-x^2}}T_k(x)T_l(x)\,\mathrm{d}x = 0 \qquad \text{if } i \ne j$$

taking an inner product with $T_n(0 \le n \le N)$, Eq. (26) gives the following linear ordinary differential equation:

$$\frac{\mathrm{d}c_{i_1}^{(k)}(t)}{\mathrm{d}t} + \sum_{m=0}^{N} P_{i_1,m} c_{i_1}^{(k)}(t) + l_n + a_n(t) = 0, \qquad 0 \le i_1 \le N$$

From Eq. (30) the boundary conditions are given by

$$c_0 = \frac{c}{2}x_f^2 + \frac{c}{4} - b^{(k-1)}x_f, \qquad c_1 = b^{(k-1)} + cx_f, \qquad c_2 = \frac{c}{4}$$
$$c_{i_1} = 0, \qquad i_1 \ne 0, 1, 2$$

For large n, calculation of the inner product involves computationally large efforts. However, using the relation equations (B5) and (B8), Chebyshev coefficients can be obtained without integration in Eq. (37), except $R(\boldsymbol{x})$. This is the main advantage of using Chebyshev polynomial to solve our problem. Finally the initial momenta are found as the optimal control Eq. (31) by algebraic manipulation of $c_{i_1,\cdots,i_n}^{(k)}(0)$ using Eq. (B5).

IV. Examples

A. Circular Orbit

Consider the two-dimensional motion described by the Hamiltonian

$$H(r, \theta, p_r, p_\theta, t) = \frac{1}{2}\left(p_r^2 + \frac{p_\theta^2}{r^2}\right) + U(r) \tag{44}$$

with $U(r) = -\frac{1}{r}$. The solution to the two-point boundary-value problem for the specific boundary condition $\boldsymbol{r}_0 = [r_0, \theta_0] = [1.0, 0.0]$, $\boldsymbol{r}_f = [r_f, \theta_f] = [1.0, 2.0]$ in $\pi/2$ units of time is sought. The canonical equations are given by

$$\dot{r} = \frac{\partial H}{\partial p_r} = p_r \tag{45}$$

$$\dot{\theta} = \frac{\partial H}{\partial p_r} = \frac{p_\theta}{r^2} \tag{46}$$

$$\dot{p}_r = -\frac{\partial H}{\partial p_r} = \frac{p_\theta^2}{r^3} - \frac{1}{r^2} \tag{47}$$

$$\dot{p}_\theta = -\frac{\partial H}{\partial p_r} = 0 \tag{48}$$

The system Eqs. (45–48) can be viewed as the following affine nonlinear system

$$\dot{\boldsymbol{x}} = g(\boldsymbol{x})\boldsymbol{u} \tag{49}$$

where

$$\boldsymbol{x} = \begin{bmatrix} r \\ \theta \end{bmatrix}, \qquad \boldsymbol{u} = \begin{bmatrix} p_r \\ \frac{p_\theta}{r} \end{bmatrix}, \qquad g(\boldsymbol{x}) = \begin{bmatrix} 1 & 0 \\ 0 & \frac{1}{r} \end{bmatrix}$$

where the control $\boldsymbol{u}(t)$ is determined to minimize the performance index

$$J = \int_{t_0}^{t_f} \frac{1}{2}\|u\|^2 + R(\boldsymbol{x})\,\mathrm{d}t \tag{50}$$

$$= \int_{t_0}^{t_f} \frac{1}{2}\left(|p_r|^2 + \frac{|p_\theta|^2}{r^2}\right) - U(r)\,\mathrm{d}t \tag{51}$$

The HJB equation is given by

$$\frac{\partial V^*}{\partial t} - U(\boldsymbol{x}) - \frac{\partial V^{*T}}{\partial \boldsymbol{x}} g(\boldsymbol{x})g(\boldsymbol{x})^T \frac{\partial V^*}{\partial \boldsymbol{x}} = 0 \tag{52}$$

Then, the GHJB equation becomes

$$\frac{\partial V^{(k)}}{\partial t} + \frac{\partial V^{(k)T}}{\partial \boldsymbol{x}}\dot{\boldsymbol{x}} - U(\boldsymbol{x}) + \frac{\|\boldsymbol{u}\|^2}{2} = 0 \tag{53}$$

$$\Leftrightarrow \frac{\partial V^{(k)}}{\partial t} + \frac{\partial V^{(k)}}{\partial r} p_r^{(k-1)} + \frac{\partial V^{(k)}}{\partial \theta}\frac{p_\theta^{(k-1)}}{r^2} - U(r) + \frac{1}{2}\left(|p_r^{(k-1)}|^2 + \frac{|p_\theta^{(k-1)}|^2}{r^2}\right) = 0 \tag{54}$$

Then, the solutions to TPVBP are given by

$$p_r^{(k+1)} = u_r^{(k+1)} = -\frac{\partial V^{(k)}}{\partial r} \tag{55}$$

$$p_\theta^{(k+1)} = r u_\theta^{(k+1)} = -\frac{\partial V^{(k)}}{\partial \theta} \tag{56}$$

For the boundary condition $\boldsymbol{r}_0 = [r_0, \theta_0] = [1.0, 0.0]$, $\boldsymbol{r}_f = [r_f, \theta_f] = [1.0, \pi/2.0]$ in $\pi/2$ units of time, the solution is a circular orbit and the initial and final velocities are given, respectively, by $[\dot{r}_0, \dot{\theta}_0] = [\dot{r}_f, \dot{\theta}_f] = [0.0, 1.0]$. Here the solution to the GHJB Eq. (56) is found using the Chebyshev expansion of V with order $N = 4$. Note that since Chebyshev polynomials are defined in $[-1, 1]$, the state variables r, θ should be rescaled to $\tilde{r}, \tilde{\theta} \in [-1, 1]$. The convergence criterion $|r_f - r_f^{(k)}| < \epsilon$ and $|\theta_f - \theta_f^{(k)}| < \epsilon$ with $\epsilon = 10^{-11}$ are adopted where $r_f^{(k)}$ and $\theta_f^{(k)}$ are the values of $r(t_f)$ and $\theta(t_f)$ found by $V^{(k)}$. Figure 1 shows the resulting trajectory in the inertial frame. Figure 2 shows the trajectories of r, θ, p_r and p_θ. The approximation error of the initial velocity found by the proposed method was less than 10^{-5}. In this example, the approximate solution of V converges after 150 iterations. The algorithm was also tested on randomly generated problems with normalized values of r, θ between zero and 1 and convergence was achieved in 150 iterations or less.

Next, consider the Hamiltonian with perturbed potential function $U(r) = -\frac{1}{r} - \frac{0.1}{r^3}$, where the higher-order term is exaggerated and it has the same TPBVP. Then, the HJB and GHJB equations with the additional term $\frac{0.1}{r^3}$ are considered and the solution to TPVBP with the same procedure is sought. The dotted line (trajectory 2) in Fig. 3 shows the trajectory using the initial condition for the unperturbed system $[\dot{r}_0, \dot{\theta}_0] = [0.0, 1.0]$, which corresponds to the previous solution. The solid line (trajectory 1) in Fig. 3 shows the trajectory using the initial conditions found by our algorithm. Figure 4 shows

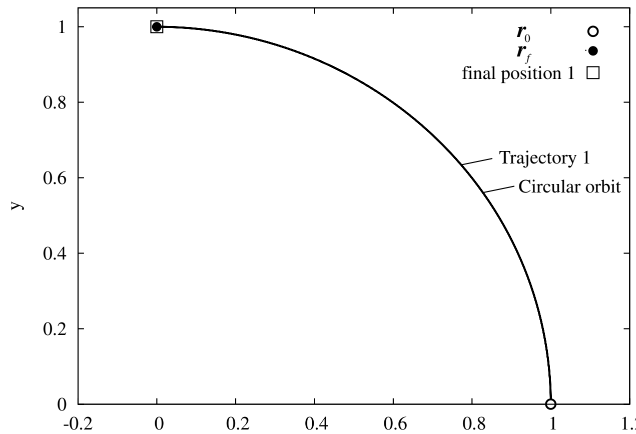

Fig. 1 Trajectory in inertial frame.

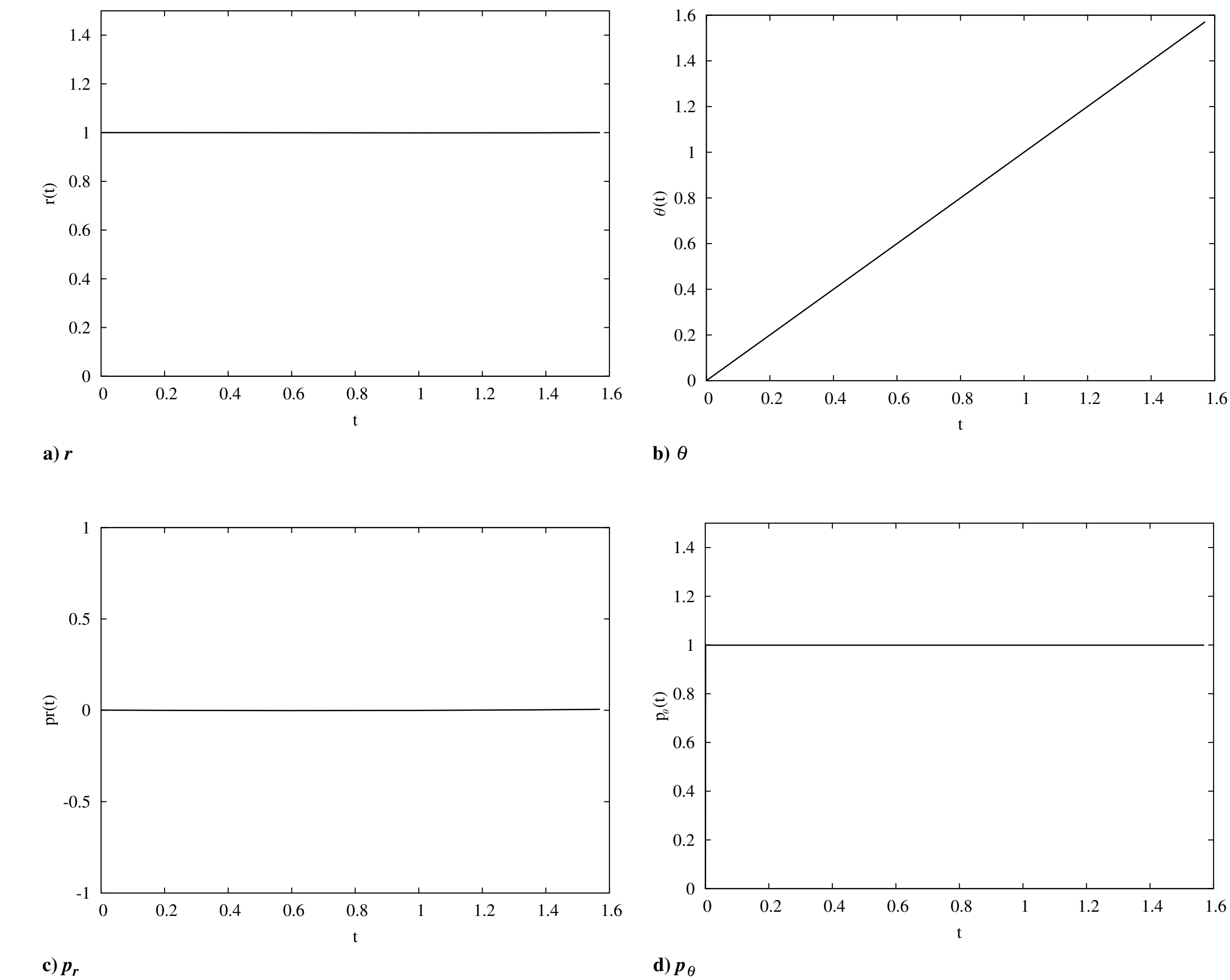

Fig. 2 Evolution of r, θ, p_r, and p_θ.

the trajectories of r, θ, p_r, and p_θ. The solution to the TPVBP was successfully found by directly solving the associated nonlinear optimal control problem. A number of the Chebyshev coefficients are shown in Tables 1 and 2.

B. Rendezvous in a Circular Restricted Three-Body Problem

In this example, a rendezvous problem to a leader spacecraft in a circular restricted three-body problem (CRTBP) is considered.

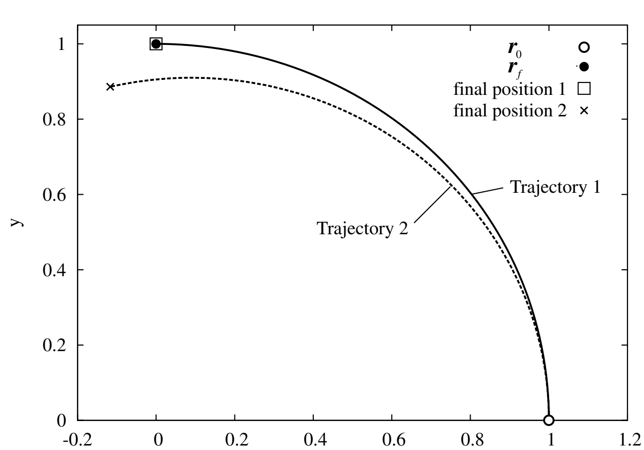

Fig. 3 Trajectory in inertial frame.

Consider the Earth–moon–spacecraft system regarded as the CRTBP, where the Earth–moon system, by assumption, rotates with a constant angular velocity ($\omega = 2.661699 \times 10^{-6}$ rad/s) about its barycenter, and the orbital motion of the two primaries is not affected by the spacecraft. Then, the equations of motion of the spacecraft in nondimensional form are given as follows [21]

$$\ddot{x} - 2\dot{y} - x = -(1-\rho)(x+\rho)/r_1^3 - \rho(x-1+\rho)/r_2^3$$
$$\ddot{y} + 2\dot{x} - y = -(1-\rho)y/r_1^3 - \rho y/r_2^3$$
$$\ddot{z} = -(1-\rho)z/r_1^3 - \rho z/r_2^3$$

$$r_1 = \sqrt{(x+\rho)^2 + y^2}, \qquad r_2 = \sqrt{(x-1+\rho)^2 + y^2}$$

where (x, y, z) is the rotating coordinate system with origin at the barycenter, z is the rotating axis, x in the direction of the Earth, y is defined to form a right-handed system, $\rho = 0.01215$ is the ratio of the mass of the moon and the total mass of the Earth and moon, and r_1 and r_2 denote the distances of the spacecraft from the Earth and moon, respectively. A unit of time is $2\pi/\omega = 27.3$ days, a unit of distance is $D = 384,748$ km, and a unit of thrust is $D\omega^2$. For the free system, there are five equilibrium points $L_1(0.83692, 0, 0)$, $L_2(1.15568, 0, 0)$, $L_3(-1.00506, 0, 0)$, $L_4(-0.48785, \sqrt{3}/2, 0)$, and $L_5(-0.48785, -\sqrt{3}/2, 0)$, known as Lagrangian points. Here, we consider the planer motion where z is set to zero and the transfer problem in the vicinity of the L_2 point, for which the velocity will steer a spacecraft to the L_2 point by an impulsive control is sought. To

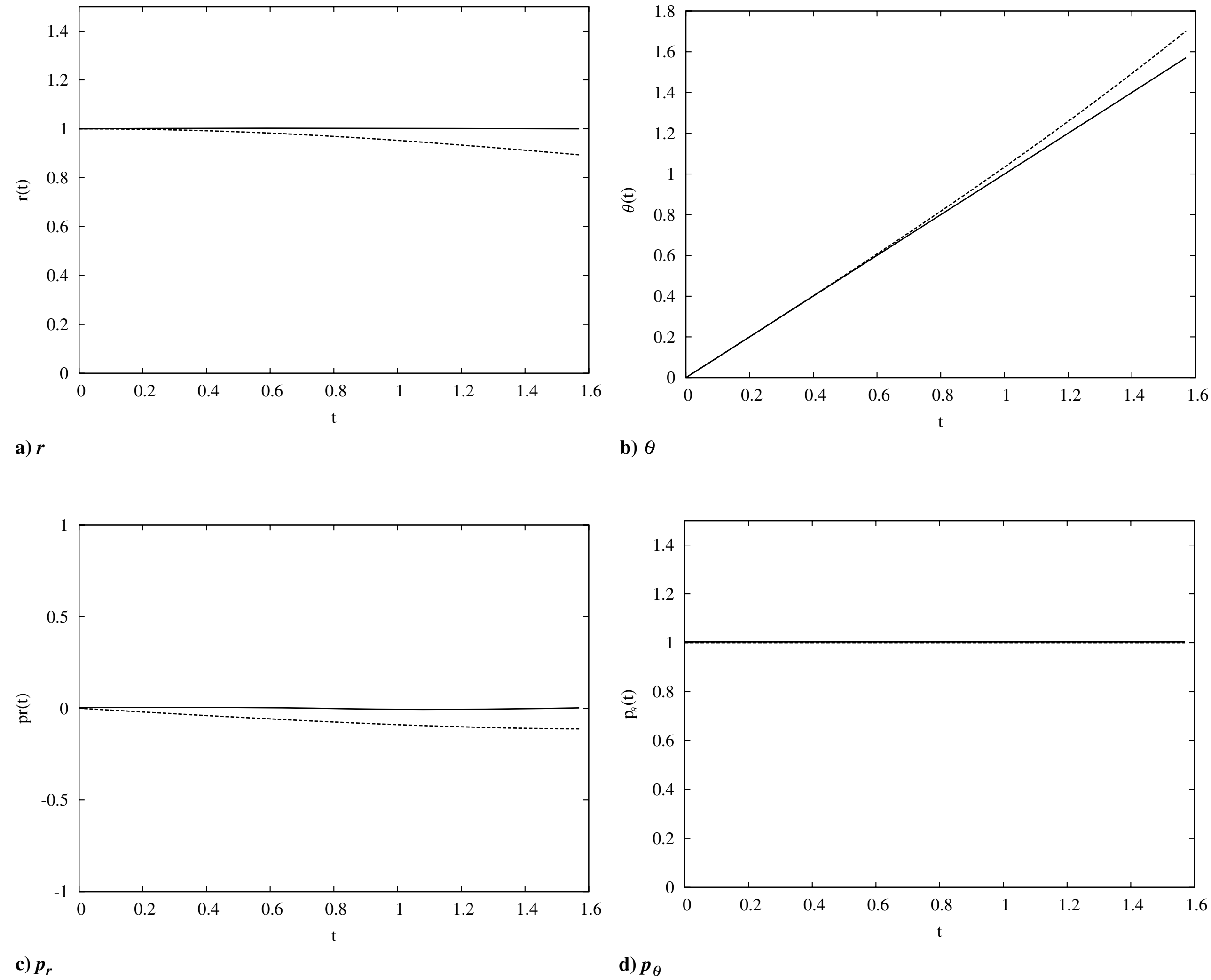

Fig. 4 Evolution of r, θ, p_r, and p_θ.

solve this problem, the following Lagrangian and Hamiltonian are considered:

$$L(x,y,\dot{x},\dot{y}) = \frac{1}{2}[(\dot{x}-y)^2 + (\dot{y}+x)^2] + \frac{1-\rho}{r_1} + \frac{\rho}{r_2} \tag{57}$$

$$p_x = \frac{\partial L}{\partial \dot{x}} = \dot{x} - y \tag{58}$$

$$p_y = \frac{\partial L}{\partial \dot{y}} = \dot{y} + x \tag{59}$$

$$H(x,y,p_x,p_y) = [\,\dot{x} \quad \dot{y}\,]\begin{bmatrix} p_x \\ p_y \end{bmatrix} - \frac{1}{2}(p_x^2 + p_y^2) - \frac{1-\rho}{r_1} - \frac{\rho}{r_2} \tag{60}$$

$$= \frac{1}{2}(p_x^2 + p_y^2) + (p_x y - p_y x) - \frac{1-\rho}{r_1} - \frac{\rho}{r_2} \tag{61}$$

Then, the canonical equations are given by

$$\dot{x} = \frac{\partial H}{\partial p_x} = p_x + y \tag{62}$$

$$\dot{y} = \frac{\partial H}{\partial p_y} = p_y - x \tag{63}$$

$$\dot{p}_x = -\frac{\partial H}{\partial x} = p_y - (1-\rho)\frac{x+\rho}{r_1^3} - \rho\frac{x-1+\rho}{r_2^3} \tag{64}$$

Table 1 Chebyshev coeffiecients $c_{n_i,n_j}(t)$.

Time	c_{00}	c_{01}	c_{02}	c_{03}	c_{10}	c_{11}	c_{12}	c_{20}
0.0	2.3901	−0.00006	0.00933	0.00001	−3.1395	−0.00120	0.00003	0.02464
1.57	1.6323	−0.00052	0.02500	0.00000	−3.1896	0.00000	0.00000	0.02500

Table 2 Chebyshev coeffiecients $c_{n_i,n_j}(t)$ for perturbed motion

Time	c_{00}	c_{01}	c_{02}	c_{03}	c_{10}	c_{11}	c_{12}	c_{20}
0.0	2.4058	−0.00036	0.00935	0.00001	−3.1516	−0.00121	0.00003	0.02463
1.57	1.6385	−0.00022	0.02500	0.00000	−3.2020	0.00000	0.00000	0.02500

$$\dot{p}_y = -\frac{\partial H}{\partial y} = -p_x - (1-\rho)\frac{y}{r_1^3} - \rho\frac{y}{r_2^3} \tag{65}$$

The system Eqs. (62–65) can be viewed as the following linear system:

$$\dot{\boldsymbol{x}} = A\boldsymbol{x} + B\boldsymbol{u} \tag{66}$$

$$\boldsymbol{x} = \begin{bmatrix} x \\ y \end{bmatrix}, \qquad \boldsymbol{u} = \begin{bmatrix} p_x \\ p_y \end{bmatrix} \tag{67}$$

$$A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \qquad B = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \tag{68}$$

For this system, an optimal control problem associated with the following performance index is considered

$$J = \int_{t_0}^{t_f} \frac{1}{2}\|u\|^2 + R(\boldsymbol{x})\,\mathrm{d}t \tag{69}$$

$$= \int_{t_0}^{t_f} \frac{1}{2}(|p_x|^2 + |p_y|^2) - U(x,y)\,\mathrm{d}t \tag{70}$$

where $U(x,y) = -\frac{1-\rho}{r_1} - \frac{\rho}{r_2}$. Then, the HJB equation becomes

$$\frac{\partial V^*}{\partial t} + \frac{\partial V^{*T}}{\partial \boldsymbol{x}} A\boldsymbol{x} - U(\boldsymbol{x}) - \frac{\partial V^{*T}}{\partial \boldsymbol{x}} BB^T \frac{\partial V^*}{\partial \boldsymbol{x}} = 0 \tag{71}$$

The GHJB equation is given by

$$\frac{\partial V^{(k)}}{\partial t} + \frac{\partial V^{(k)T}}{\partial \boldsymbol{x}}\dot{\boldsymbol{x}} + \frac{\|\boldsymbol{u}\|^2}{2} - U(\boldsymbol{x}) = 0 \tag{72}$$

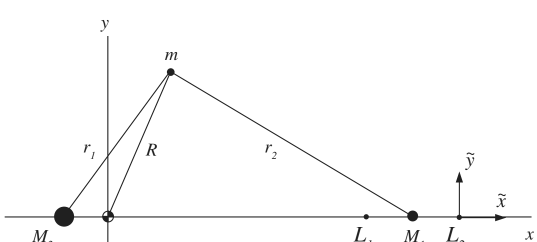

Fig. 5 Lagrangian points.

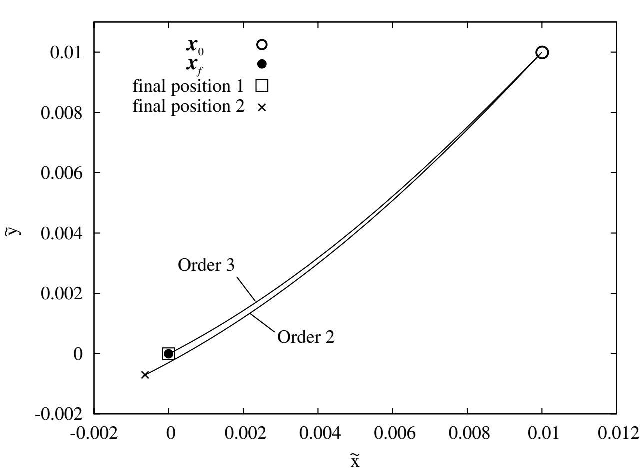

Fig. 6 Trajectory in rotating frame.

$$\Leftrightarrow \frac{\partial V^{(k)}}{\partial t} + \frac{\partial V^{(k)}}{\partial x}y - \frac{\partial V^{(k)}}{\partial y}x + \frac{\partial V^{(k)}}{\partial x}p_x^{(k-1)} + \frac{\partial V^{(k)}}{\partial y}p_y^{(k-1)} + \frac{1}{2}(|p_x^{(k-1)}|^2 + |p_y^{(k-1)}|^2) - \frac{1-\rho}{r_1} - \frac{\rho}{r_2} = 0 \tag{73}$$

$$V^{(k)}(\boldsymbol{p}_f, \boldsymbol{x}, t_f) = -\boldsymbol{p}_f^T\boldsymbol{q} = -p_{x_f}x - p_{y_f}y \tag{74}$$

and the solutions to TPVBP are given by

$$p_x^{(k+1)} = -\frac{\partial V^{(k)}}{\partial x} \tag{75}$$

$$p_y^{(k+1)} = -\frac{\partial V^{(k)}}{\partial y} \tag{76}$$

For numerical implementation, the origin of the coordinate system (x,y,z) is shifted to $L_2(1.15568,0,0)$ (see Fig. 5) and $U(x,y)$ is shifted to assure that $U(x,y) < 0$ in the domain of interest. Figure 6 shows the trajctory of the Chebyshev polynomials of order $N = 2$ and 3. The trajectory that satisfies the boundary condition is successfully obtained for $N = 3$.

V. Conclusions

A method to approximate the solution of the Hamilton–Jacobi equation that can solve Lambert's problem is proposed. Successive approximation and the Galerkin spectral method have been applied to the solution of Lambert's problem. As a major advantage, the proposed algorithm can accommodate perturbations without any modification when the Lagrangian function is expressed in terms of the kinetic energy and the potential function that satisfies $U(r) < 0$. Moreover, the higher-order terms are easily computed because our algorithm employs the Chebyshev polynomials. Although the main idea behind the method is rather simple, the transformed optimal control problem is so general that other various optimal control design methods can also be applied to the Lambert's problem.

Appendix A: Monotonicity Conditions for *V*

Following [16], we will show that the corresponding value function $V^{(k)}$ satisfies the inequality

$$V^*(\boldsymbol{x},t) \le V^{(k)}(\boldsymbol{x},t) \le V^{(k-1)}(\boldsymbol{x},t) \tag{A1}$$

Note that Eq. (A1) implies that $V^{(k)}$ is monotonically decreasing and bounded by $V^*(\boldsymbol{x},t)$ and hence

$$\lim_{k\to\infty} V^{(k)}(\boldsymbol{x},t) = V^*(\boldsymbol{x},t) \tag{A2}$$

$$u^* = -g(\boldsymbol{x})^T\frac{\partial V^*}{\partial x} \tag{A3}$$

for x, $t \in [t_0, t_f]$.

Set

$$V^{(k)}(\boldsymbol{x}, \boldsymbol{u}^{(k+1)}) = V^{(k-1)}(\boldsymbol{x}, \boldsymbol{u}^{(k)}) + \Delta V(\boldsymbol{x}, \boldsymbol{u}^{(k)}) \tag{A4}$$

Substituting Eq. (A4) into Eq. (26) yields

$$\frac{\partial V^{(k-1)}}{\partial t} + \frac{\partial V^{(k-1)T}}{\partial \boldsymbol{x}}f(\boldsymbol{x}) - \frac{1}{2}\frac{\partial V^{(k-1)T}}{\partial \boldsymbol{x}}g(\boldsymbol{x})g(\boldsymbol{x})^T\frac{\partial V^{(k-1)}}{\partial \boldsymbol{x}} + R(\boldsymbol{x}) = -\frac{\partial \Delta V}{\partial t} - \frac{\partial \Delta V^T}{\partial \boldsymbol{x}}f(\boldsymbol{x}) - \frac{\partial V^{(k-1)T}}{\partial \boldsymbol{x}}g(\boldsymbol{x})g(\boldsymbol{x})^T\frac{\partial \Delta V}{\partial \boldsymbol{x}} \tag{A5}$$

$$= -\frac{\mathrm{d}\Delta V}{\mathrm{d}t} \tag{A6}$$

Also substituting $V^{(k-1)} = V^{(k)} - \Delta V$ into Eq. (26) yields

$$\frac{\partial V^{(k)}}{\partial t}+\frac{\partial V^{(k)T}}{\partial \boldsymbol{x}}f(\boldsymbol{x})-\frac{\partial V^{(k)T}}{\partial \boldsymbol{x}}g(\boldsymbol{x})g(\boldsymbol{x})^T\frac{\partial (V^{(k)}-\Delta V)}{\partial \boldsymbol{x}}+R(\boldsymbol{x})+\frac{1}{2}\frac{\partial (V^{(k)}-\Delta V)^T}{\partial \boldsymbol{x}}g(\boldsymbol{x})g(\boldsymbol{x})^T\frac{\partial (V^{(k)}-\Delta V)}{\partial \boldsymbol{x}}=0 \tag{A7}$$

Then

$$\frac{\partial V^{(k)}}{\partial t}+\frac{\partial V^{(k)T}}{\partial \boldsymbol{x}}f(\boldsymbol{x})-\frac{1}{2}\frac{\partial V^{(k)T}}{\partial \boldsymbol{x}}g(\boldsymbol{x})g(\boldsymbol{x})^T\frac{\partial V^{(k)}}{\partial \boldsymbol{x}}+R(\boldsymbol{x})=-\frac{1}{2}\|g(\boldsymbol{x})^T\frac{\partial \Delta V}{\partial \boldsymbol{x}}\|^2\leq 0 \tag{A8}$$

Hence, from Eqs. (A6) and (A8)

$$\frac{\mathrm{d}\Delta V}{\mathrm{d}t}=\frac{1}{2}\|g(\boldsymbol{x})^T\frac{\partial \Delta V}{\partial \boldsymbol{x}}\|^2\geq 0 \tag{A9}$$

From the boundary conditions

$$V^{(k)}(t_f)=\phi^{(k)}(\boldsymbol{x}(t_f)),\qquad V^{(k+1)}(t_f)=\phi^{(k+1)}(\boldsymbol{x}(t_f)) \tag{A10}$$

and Eq. (33)

$$\Delta V(t_f)=V^{(k+1)}(t_f)-V^{(k)}(t_f) \tag{A11}$$

$$=\phi^{(k+1)}(\boldsymbol{x}(t_f))-\phi^{(k)}(\boldsymbol{x}(t_f))\leq 0 \tag{A12}$$

Hence, Eqs. (A9) and (A12) imply

$$\Delta V(t)=V^{(k+1)}-V^{(k)}\leq 0 \tag{98}$$

for all $t\in[t_0,t_f]$.

Appendix B: Relation of Chebyshev Coefficients

The Chebyshev polynomials have the following relations:

$$T_i(x)T_j(x)=\frac{1}{2}(T_{i+j}(x)+T_{|i-j|}(x)) \tag{B1}$$

$$2T_k(x)=\frac{1}{k+1}\frac{\mathrm{d}T_{k+1}(x)}{\mathrm{d}x}-\frac{1}{k-1}\frac{\mathrm{d}T_{k-1}(x)}{\mathrm{d}x} \tag{B2}$$

Using Eqs. (B1) and (B2), Chebyshev coefficients can be obtained without integration in Eq. (37) except for the potential term $U(x)$. Consider an approximating expansion of the form

$$f(x)=\sum_{n=0}^{N}a_nT_n(x),\qquad g(x)=\sum_{n=0}^{N}b_nT_n(x) \tag{B3}$$

Let the derivative of $f(x)$ be

$$f'(x)\triangleq\sum_{n=0}^{N}a'_nT_n(x) \tag{B4}$$

Then, the corresponding coefficients are given by

$$a'_n=\begin{cases}\sum_{j=n+1,j-n:\ \mathrm{odd}}^{N}2ja_j & (n\geq 1)\\ \sum_{j=1,j-n:\ \mathrm{odd}}^{N}ja_j & (n=0)\end{cases} \tag{B5}$$

Also, let the product of $f(x)$ and $g(x)$ be

$$f(x)g(x)=\left(\sum_{n=0}^{N}a_nT_n(x)\right)\left(\sum_{n=0}^{N}b_nT_n(x)\right) \tag{B6}$$

$$\triangleq\sum_{n=0}^{N}c_nT_n(x) \tag{B7}$$

and the corresponding coefficients are

$$c_n=\frac{1}{2}\sum_{i,j=0}^{N}\sum_{i+j=n,|i-j|=n}^{N}a_ib_j \tag{B8}$$

The generalization to more than two variables is straightforward. Using Eqs. (B5) and (B8), Chebyshev coefficients can be obtained without integration.

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